

Intersecting Connes Noncommutative Geometry with Quantum Gravity

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Abstract

An intersection of Noncommutative Geometry and Loop Quantum Gravity is proposed. Alain Connes' Noncommutative Geometry provides a framework in which the Standard Model of particle physics coupled to general relativity is formulated as a unified, gravitational theory. However, to this day no quantization procedure compatible with this framework is known. In this paper we consider the noncommutative algebra of holonomy loops on a functional space of certain spin-connections. The construction of a spectral triple is outlined and ideas on interpretation and classical limit are presented.

1 Introduction

The framework of noncommutative geometry [1] suggests an appealing solution to one of the central riddles of theoretical physics, the unification of general relativity and the standard model. However, the noncommutative formulation of the standard model is intrinsically classical and no notion of quantization within this framework is known. In this paper we attempt to address this problem by suggesting an intersection of noncommutative geometry with principles of Loop Quantum Gravity (LQG): The idea presented is to apply the machinery of noncommutative geometry to the algebra of holonomy loops. This algebra is naturally noncommutative and stores topological information about an underlying space of connections. The goal is a spectral triple over this functional space of geometries.

In the following we will outline and clarify ideas already presented in [2]. Emphasis is put on the general idea rather than technical details. First, in section 2, we briefly introduce the standard model framed within noncommutative geometry and propose an application of noncommutative geometry to a functional space of Euclidean gravity. In section 3 we outline the construction of a spectral triple and, in section 4, an interpretation of the construction is presented. Finally we conclude and discuss various problems in section 5.

2 Noncommutative geometry, the standard model and quantization

A noncommutative geometry in the sense of Connes is determined by a spectral triple $(\mathcal{B}, D, \mathcal{H})$ which consist of a $*$ -algebra \mathcal{B} represented on a Hilbert space \mathcal{H} on which a self-adjoint unbounded operator D , the Dirac operator, acts. The triple is normally required to satisfy a set of seven axioms proposed by Connes [3]. Ordinary Riemannian spin-geometries form a subset in this framework and are described by commutative C^* -algebras. Here, the underlying manifold \mathcal{M} emerges as the spectrum of the algebra and a differential structure is provided by the Dirac operator. For example, the distance between points $x, y \in \mathcal{M}$ can be formulated algebraically [3]

$$d(x, y) = \sup_{f \in \mathcal{B}} \left\{ |\chi_x(f) - \chi_y(f)| \mid \| [D, f] \| \leq 1 \right\}, \quad (1)$$

where $\|\cdot\|$ on the rhs is the supremum norm and χ_x is the character corresponding to the point x , i.e. $\chi_x(f) = f(x)$. Further, we can recover the Clifford algebra, and so differential forms, by considering commutators of the Dirac operator

$$\xi = f_1[D, f_2] \dots [D, f_k], \quad f_i \in \mathcal{B}. \quad (2)$$

Differential structures such as (1) and (2) continue to make sense even when the algebra is noncommutative. Such algebras can not always be identified as function algebras over manifolds and the set of points, the spectrum of the algebra, is often reduced and may, although the algebra is infinite dimensional, be discrete. However, noncommutative geometry permits differential structures which treats this broad variety of spaces on an equal footing.

The Standard Model

A special class of noncommutative geometries are the *almost commutative geometries* described by spectral triples of the form

$$\begin{aligned} \mathcal{B} &= C^\infty(\mathcal{M}) \otimes \mathcal{B}_F, \\ D &= \not{D} \otimes \gamma^d + 1 \otimes D_F, \\ \mathcal{H} &= L^2(\mathcal{M}, S \times M_n(\mathbb{C})), \end{aligned} \quad (3)$$

where \mathcal{B}_F is a finite dimensional matrix algebra, \not{D} is the usual space-time Dirac operator and D_F is a matrix operator satisfying certain criteria. The dimension of the (Riemannian, compact) manifold \mathcal{M} and $\gamma^d = \gamma^1 \cdot \dots \cdot \gamma^{d-1}$ are the gamma matrices. The spinor-bundle is denoted by S . The almost commutative geometry given by

$$\mathcal{B}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad (4)$$

where \mathbb{H} denotes quaternions, is the algebra which forms the basis of the formulation of the standard model in terms of noncommutative geometry [4, 3]. Here the Hilbert space is the Hilbert space of the entire fermionic content of the standard model and D_F contains information about the fermion masses. It is natural to consider fluctuations around the Dirac operator of the form

$$D \rightarrow \tilde{D} = D + A + JAJ^\dagger, \quad (5)$$

where J denotes charge conjugation and

$$A = \sum_i a_i [D, b_i] , \quad a_i, b_i \in \mathcal{B} , \quad (6)$$

is a general one-form in the sense of (2). Combining (6) with (4) one derives the entire bosonic sector of the standard model with the Higgs boson emerging as an integrated part of the noncommutative gauge field.

The algebraic equivalence of diffeomorphism invariance is invariance under automorphisms of the algebra. For noncommutative algebras like (3) the automorphism group is larger than the diffeomorphism group, including also inner automorphisms, i.e. gauge transformations. Requiring invariance under automorphisms of the algebra leads to the spectral action principle which states that physics only depends on the spectrum of the Dirac operator. This statement is quantified by the action formula [5, 6]

$$S = \langle \xi | \tilde{D} | \xi \rangle + \text{Trace } \varphi \left(\frac{\tilde{D}}{\Lambda} \right) , \quad (7)$$

where ξ is a Hilbert state and φ a suitable function selecting eigenvalues below the cutoff Λ . Equation (7) contains the entire classical action of the standard model coupled to gravity.

It is intriguing that Yang-Mills-Higgs models compatible with Connes noncommutative geometry appear to be rather rare [7, 8, 9, 10].

To conclude, the standard model coupled to gravity can be formulated as pure geometry over a 'space' characterized by an almost commutative algebra. Fermions are linked to the metric structure and the bosonic sector arises through fluctuations around the Dirac operator, the 'metric' of noncommutative geometry.

Quantization

The fact that the standard model can be formulated as a gravitational theory suggests that it can not be quantized in any straightforward manner within the framework of noncommutative geometry since such a quantization would, accordingly, involve quantum gravity. On the other hand, this implies that the search for a suitable quantization scheme might pass through quantum gravity. This is the problem which we wish to address in this paper: How principles of noncommutative geometry could be unified with ideas in quantum gravity.

2.1 Intersecting noncommutative geometry and quantum gravity

Consider first quantum field theory. It involves, via Feynman path integrals, integration theory over spaces of field configurations. The central object is the partition function, the generating functional for Greens functions

$$\mathcal{Z}[J] = \int [d\Phi] \exp(-\frac{i}{\hbar} S[\Phi, J]) ,$$

where Φ denotes the field content of the model characterized by the classical action $S[\Phi]$ coupled to external fields J . We now propose the following: Since Connes formulation of the standard model lacks a clear quantization procedure and since quantum field theory deals with integration theory over spaces of field configurations, it seems natural to try to apply the machinery of noncommutative geometry to functional spaces. Further, since Connes formulation of the standard model is essentially a gravitational theory we suggest investigating a configuration space related to gravity.

In fact, a configuration space suitable for our purposes has already been investigated in the literature. Loop Quantum Gravity (LQG) [11, 12, 13] is an attempt to quantize general relativity using methods of canonical quantization. The configuration space relevant for LQG is a space \mathcal{A} of $SU(2)$ connections which are interpreted as certain spin-connections living on a 3-dimensional hyper-surface. This surface emerges from a foliation of 4-dimensional space-time which is needed for the quantization procedure.

Central to LQG is an algebra of Wilson loops $W(L)$ which form an abelian algebra of observables on the space of connections

$$\begin{aligned} W(L) : \mathcal{A} &\rightarrow \mathbb{C} , \\ \nabla &\rightarrow Tr \text{ Hol}(L, \nabla) , \end{aligned} \tag{8}$$

where $\text{Hol}(L, \nabla)$ is the holonomy of the connection ∇ along the loop L and Tr is the trace with respect to the representation of the group. One of the advantages of this formulation is that it permits a natural implementation of diffeomorphism invariance in a way that leads to a countable structure, including a separable Hilbert [14]: Roughly, the set of Wilson loops form certain labeled, oriented graphs of increasing complexity and, up to diffeomorphisms, only the structure of graphs is relevant. This structure is countable¹.

¹In fact, it turns out that so-called extended diffeomorphisms are required to obtain countable structures. Extended diffeomorphisms permit finitely many non-smooth points.

We believe that there exist a natural intersection between LQG and non-commutative geometry: instead of using Wilson loops we suggest to study the noncommutative algebra of holonomy loops themselves. By avoiding the trace the gauge symmetry of local Lorentz transformations is preserved. Further, since the objective is to apply the machinery of noncommutative geometry to the functional space, rather than a canonical quantization procedure, we propose to consider space-time as a whole and avoid a foliation. Thus, we consider an algebra of space-time holonomy loops

$$\begin{aligned} L : \mathcal{A} &\rightarrow G, \\ \nabla &\rightarrow \text{Hol}(L, \nabla), \end{aligned} \tag{9}$$

where G is the symmetry group. Since compactness of the gauge group is at present needed for the analysis, we are at first limited to consider Euclidean gravity with, for example,

$$G = SO(4).$$

Finally, rather than postulating constraints on the Hilbert space, such as the Hamilton constraint in LQG, we suggest to apply the spectral action principle [5, 6]. This amounts to seek physical information in the spectrum of the Dirac operator.

This intersection of LQG and noncommutative geometry contains all the ingredients we are looking for: Integration theory over a functional space related to gravity² involving a natural noncommutative algebra.

The holonomy loops are matrix valued and can, as we will show below, be heuristically argued to entail an almost commutative algebra in a classical limit characterized by a single space-time geometry, that is, a single point in \mathcal{A} . This is encouraging and provides us with the hope that low energy physics characterized by an almost commutative algebra may arise as the classical limit of a pure quantum gravity.

3 The construction

Let us go into some details. As already mentioned, the space of interest is the space \mathcal{A} of connections with values in the Lie-algebra of a compact group G . As algebra of observables it is natural to take functions on \mathcal{A} . A natural

²Since the algebra of holonomy loops lives on a space of connections the corresponding Hilbert space will be equiped with an inner product involving a functional integration [2].

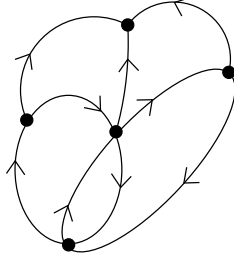


Figure 1: A generic graph.

collection of functions on \mathcal{A} is the traced holonomies i.e. given a closed loop L the associated function is

$$Tr \circ f_L : \mathcal{A} \ni \nabla \rightarrow Tr(Hol(L, \nabla)) .$$

It can be shown that this is a complete set of functions, i.e. that the algebra of linear combinations of these functions completely determines \mathcal{A} modulo gauge equivalence [15].

Since we are interested in a noncommutative algebra we will take untraced holonomies and therefore get functions over \mathcal{A} with values in the matrix representation of G . The algebra we therefore want to consider is the algebra of all linear combinations of such functions i.e. functions on the form

$$a_1 f_{L_1} + \dots + a_n f_{L_n} .$$

This algebra comes with a norm, namely the usual sup-norm over \mathcal{A} . The completion in this norm will be a C^* -algebra.

Similar to LQG this C^* -algebra can be seen as matrix valued functions over a space $\bar{\mathcal{A}}$ containing \mathcal{A} as a dense subset and $\bar{\mathcal{A}}$ can be written as $Hom(\mathcal{L}_{x_0}, G)$ where \mathcal{L}_{x_0} is the loop group based in a given point x_0 , i.e. loops modulo trivial backtracking and reparameterization, the group structure being composition of loops.

3.1 Inductive systems and geometrical structures

Construction of geometrical structures directly on \mathcal{A} does not seem easy. Instead we will start by looking at \mathcal{A} as "seen from" a finite collection of loops, L_1, \dots, L_n . This can be interpreted as a regularization of the functional

space \mathcal{A} . Seen from this collection of loops \mathcal{A} just looks like G^n : Namely, a connection $\nabla \in \mathcal{A}$ gives rise to an element in G^n via

$$(Hol(L_1, \nabla), \dots, Hol(L_n, \nabla)) .$$

Therefore, in this regularized picture the *functional space* \mathcal{A} can be identified with a manifold³. Thus, a connection is a point (g_1, \dots, g_n) on G^n . Each value g_i should be thought of as the holonomy along the i 'th loop, L_i . On G^n it is easy to construct various structures. If we for example want to construct a Hilbert space, it is natural to take $L^2(G^n)$, the space of square integrable functions on G^n with respect to the Haar measure on G^n . These are functions on the connections restricted to the graph, denoted Γ , spanned by L_1, \dots, L_n . The inner product on this Hilbert space should be interpreted as a functional integral over connections and any derivations acting on this space as functional derivations. In general, loops may intersect and the corresponding graph will consist of a number of edges and vertices, see fig.1. In this case the number of edges, n , corresponds to the number of independent line segments and the corresponding space will be G^n (see [15] or [17] for details). If we have another collection of loops L'_1, \dots, L'_m whose graph Γ' contains the graph Γ the functions living on the graph Γ can be considered as functions living on Γ' . Formally we get an embedding of Hilbert spaces $L^2(G^n)$ into $L^2(G^m)$, see fig.2.

The construction of the space of square integrable functions on \mathcal{A} follows by considering functions that lies in one of the spaces $L^2(G^n)$ associated to a finite collection of loops and identify them if they coincide on some graph containing both of them. In this way we get functions on connections restricted to all graphs, i.e. functions on connections on the entire M , that is \mathcal{A} .

The formal language of the construction is that of inductive limits of Hilbert spaces.

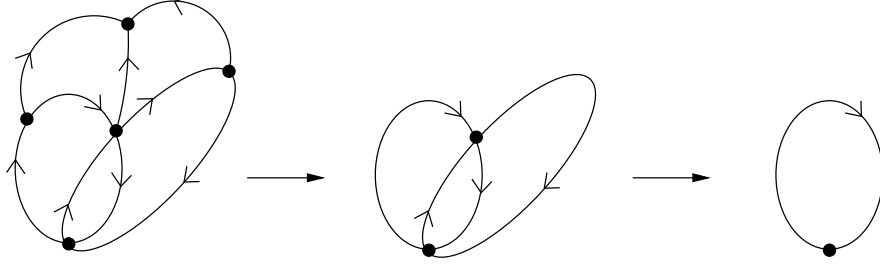


Figure 2: Part of a projective system of graphs.

3.2 The spectral triple

The aim is to construct a spectral triple on the algebra of untraced holonomies. For a finite collection of loops L_1, \dots, L_n there is an obvious Hilbert space⁴ and Dirac operator, namely $L^2(G^n, Cl(TG^n))$ and the usual Euler-Dirac operator $D = d + d^*$. By applying the same construction as in the previous section we get the Hilbert space $L^2(\bar{\mathcal{A}}, Cl(T\bar{\mathcal{A}}))$ and an Euler-Dirac operator D on this space. Here Cl denotes the Clifford algebra. By tensoring with M_N , N being the size of the representation of G , we end up with a triple

$$(A_{ut}, D \otimes 1, L^2(\bar{\mathcal{A}}, Cl(T\bar{\mathcal{A}}) \otimes M_N)) , \quad (10)$$

where the algebra A_{ut} of untraced holonomies acts pointwise on the M_N -part of $Cl(T\bar{\mathcal{A}}) \otimes M_N$. This is our spectral triple.

The triple (10) is however far from fulfilling the conditions of Connes, since the kernel of D is infinite and not even separable (recall that the Hilbert space is not separable). In the next section we will explain how the symmetry group of diffeomorphisms can be used to obtain countable structures.

Several problems arise during the construction of the triple. The key question is to construct an embedding of Hilbert spaces $L^2(G^n, Cl(TG^n))$ into $L^2(G^m, Cl(TG^m))$ compatible with the embedding of the corresponding graphs. This boils down to the construction of a metric on G^n compatible with the embeddings. In [2] we circumvented the problem by discarding intersecting loops. This, however, is clearly unsatisfactory since the inclusion

³This makes \mathcal{A} , or rather its closure $\bar{\mathcal{A}}$, a so-called *pro-manifold* since it can be identified with a projective system of manifolds. This leaves $\bar{\mathcal{A}}$ with nice properties. In particular, $\bar{\mathcal{A}}$ has a canonical topology [16].

⁴We would prefer the Hilbert space $L^2(G^n, S)$ involving spin structure on G^n . However, this entails embedding problems for which we have found no solution, see [2].

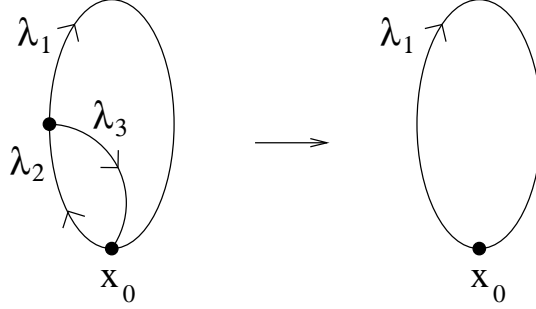


Figure 3: An $n = 3$ graphs projected down to the $n = 1$ graph.

of intersecting loops is essential in order to obtain the correct projective limit. We will address this problem in a forthcoming publication.

3.3 An $n = 3$ graph

Before we continue with diffeomorphism invariance let us use an example to clarify the construction and point out its weak point. Consider the graph Γ consisting of two vertices and three edges as shown on the lhs of fig 3. The space \mathcal{A} restricted to this graph is identified with the manifold

$$\mathcal{A}_\Gamma \simeq G^3 ,$$

which means that a connection is characterized by three group elements, $\nabla = (g_1, g_2, g_3)$. Each group element g_i is interpreted as the parallel transport of ∇ along the line segment l_i . We are interested in the noncommutative algebra of (holonomy-) loops which in this case is generated by the elements

$$L_{12} = g_1 \cdot g_2 , \quad L_{13} = g_1 \cdot g_3 , \quad L_{3*2} = g_3^{-1} \cdot g_2 . \quad (11)$$

since combined line-segments such as $l_1 \circ l_2$ form closed loops.

Next, let $\{e_i\}$ be a global basis of the tangent bundle to G^3 and $\{\bar{e}_i\}$ its dual. The Clifford bundle is constructed by imposing the anti-commutator relation $\{\bar{e}_i, \bar{e}_j\} = 2\langle \bar{e}_i | \bar{e}_j \rangle$. We construct the Hilbert space

$$\mathcal{H}_\Gamma = L^2(G^3, Cl(T^*G^3) \otimes M_N(\mathbb{C})) , \quad (12)$$

of matrix-valued functions on G^3 with additional values in the Clifford bundle. In (12) N is the size of the representation of G . The algebra of loops

(11) acts by matrix multiplication on the factor $M_N(\mathbb{C})$ in (12). For example, the loop L_{12} acts on \mathcal{H}_Γ by

$$L_{12}\Psi(\nabla) := (id \otimes \nabla(L_{12}))\Psi(\nabla) = (id \otimes g_1 \cdot g_2)\Psi(g_1, g_2, g_3) , \quad \Psi \in \mathcal{H}_\Gamma .$$

The inner product consist of three components: Integration over the group, trace of the matrices and the inner product on the Clifford bundle. Thus, if we write $\Psi = \Psi_{ij}e_{k_1} \cdots e_{k_l}$ where i, j are indices of the matrix the norm associated to the inner product reads (sum over repeated indices)

$$\langle \Psi^1 | \Psi^2 \rangle = \int d\mu \Psi_{ij}^1 (\Psi_{ij}^2)^* \langle e_{k_1}^1 \cdots e_{k_l}^1 | e_{k_1}^2 \cdots e_{k_l}^2 \rangle , \quad \Psi^i \in \mathcal{H}_\Gamma \ \forall i \in \{1, 2\} ,$$

where $d\mu$ is the Haar measure on G^3 and $(\Psi_{ij})^*$ is the complex conjugate of Ψ_{ij} . Notice that by taking the matrix trace we turn holonomy loops into gauge-invariant Wilson loops. Further, the integration over the group should, as already mentioned, be interpreted as a functional integral since each point in G^3 represents a connection; more on this later.

The Dirac operator has the form

$$D = \sum_i \bar{e}_i \nabla_{e_i} , \tag{13}$$

where ∇_{e_i} denotes the Levi-Civita connection in the direction of e_i .

As shown in fig. 3 the graph Γ is related to the graph of a single loop via the projection

$$P(g_1, g_2, g_3) = g_1 \cdot g_2 . \tag{14}$$

So far we did not specify the inner product on T^*G^3 . In fact, this turns out to be a crucial point. Let us demonstrate this by taking the simplest case, $G = U(1)$. The most obvious attempt for an inner product would be the product metric on G^3 ; i.e. the three copies of G are orthogonal. This is however not going to work for the projection (14) for the following reason: We use the coordinate θ on $U(1)$ given by $g = \exp(2\pi i\theta)$. In such coordinates the projection (14) is given by

$$P(\theta_1, \theta_2, \theta_3) = \theta_1 + \theta_2 := \theta .$$

Denote by $d\theta_i$ an orthonormal basis for the cotangent bundle T^*G^3 . The inner product on the cotangent bundle is given by

$$\langle d\theta_i | d\theta_j \rangle = \delta_{ij} .$$

We consider now the norm of the vector $d\theta_1 + d\theta_2 \in T^*G^3$

$$\langle d\theta_1 + d\theta_2 | d\theta_1 + d\theta_2 \rangle = 2 . \quad (15)$$

On the other hand, the norm of the vector $d\theta \in T^*G$ is

$$\langle d\theta | d\theta \rangle = 1 . \quad (16)$$

However, the two vectors $d\theta_1 + d\theta_2$ and $d\theta$ are related by the induced projection P^*

$$P^* : T^*G \rightarrow T^*G^3 , \quad P^*(d\theta) = d\theta_1 + d\theta_2 ,$$

which means that the inner product on the cotangent spaces is not compatible with the projection (14) since (15) does not equal (16). Thus, the Hilbert space (12) is not compatible with the projection either⁵. This means that we cannot take the inductive limit of the spectral triple candidate (11)+(12)+(13) in any consistent way.

We see two obvious strategies to solve this problem. First, one can attempt to construct an inner product on the cotangent space which is compatible with projections of the form (14). Such an inner product must necessarily leave different copies of G non-orthogonal, i.e.

$$\langle d\theta_1 | d\theta_2 \rangle \neq 0 .$$

This will change the Dirac operator since it contains the metric.

Second, one can attempt to construct a projective system which avoids projections of the form (14) altogether.

We shall not elaborate further on these difficulties here as we shall address them elsewhere.

3.4 Diffeomorphism invariance

We will now address the problem of diffeomorphism invariance (see [2] for details). The naive idea is to define the Hilbert space \mathcal{H}_{Diff} of diffeomorphism invariant states by

$$\sum_{\phi \in Diff(M)} \phi(\xi), \quad \xi \in L^2(\bar{\mathcal{A}}, Cl(T\bar{\mathcal{A}}) \otimes M_N). \quad (17)$$

⁵These difficulties were encountered also in [17] where the authors suggested a metric compatible with all projections. However, the metric constructed turns out to be degenerate.

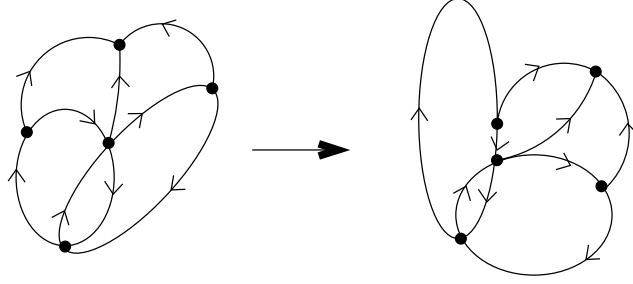


Figure 4: Two graphs related by a diffeomorphism.

This sum is manifestly diffeomorphism invariant, and of course does not make sense. However, the space $L^2(\bar{\mathcal{A}}, Cl(T\bar{\mathcal{A}}) \otimes M_N)$ is made up of functions living on graphs. If we consider a graph, it can be deformed, by a diffeomorphism, into any other graph with the same combinatorial structure (or at least by an extended diffeomorphism, see [14]). As the part of \mathcal{H}_{Diff} living on graphs with a given combinatorial structure we can therefore just take the Hilbert space of functions living on one fixed graph with this combinatorial structure. This however, would be to overlook the diffeomorphisms mapping the graph into itself. Hence as the part of \mathcal{H}_{Diff} living on graphs with a given combinatorial structure we take functions of the form (17), where ξ belongs to the Hilbert space of a fixed graph with the given combinatorial structure. This time the sum makes sense, since it is finite (it basically consist of permutations of edges and we also strictly speaking need to weight the sum).

We thus have the parts of \mathcal{H}_{Diff} corresponding to each combinatorial structure a graph can have. The construction of \mathcal{H}_{Diff} is then the same as for $L^2(\bar{\mathcal{A}}, Cl(T\bar{\mathcal{A}}) \otimes M_N)$ by considering embeddings of smaller graphs into bigger graphs. In this way we get a Hilbert space corresponding to one infinite "graph" containing only the combinatorics of graphs on the manifold.

The construction of the algebra is similar and the Dirac operator on the space of connections descends to \mathcal{H}_{Diff} since it is diffeomorphism invariant.

The Hilbert space \mathcal{H}_{Diff} is separable, since the combinatorics of graphs is countable.

It is not clear to us whether it is wise to treat diffeomorphisms as described above, or whether one should keep them as elements of the automorphism group. In any case, we find it intriguing that, up to (extended)

diffeomorphisms, only countable structures remain.

4 Discussion

Let us here discuss the interpretation of the setup:

The inner product as a path-integral

In the previous section we mentioned the Hilbert space of states identified up to diffeomorphisms. Let $\Psi \in \mathcal{H}_{Diff}$ and consider automorphism invariant quantities of the form

$$\langle \Psi | \dots | \Psi \rangle . \quad (18)$$

The inner product involves, after the appropriate limit is taken, an integration over G^∞ which is, up to diffeomorphisms, associated to the space of connections \mathcal{A} . Thus, the interpretation of (18) as a path integral of the form

$$\sim \int_{\mathcal{A}/diff} \dots \quad (19)$$

lies at hand. The object in (19) involves an integration which can be interpreted as a sum over all 'geometries' up to diffeomorphisms. One could impose a spectral action principle [5] and consider automorphism invariant quantities of the form⁶

$$\langle \Psi | D | \Psi \rangle , \quad Tr D^2 , \quad \dots$$

which might combine to some sort of an effective action of quantum gravity.

An emerging almost commutative algebra

Consider a classical connection ∇_0 in \mathcal{A} . In a classical limit a state $\Psi(\nabla)$ should peak around a classical geometry, for example ∇_0 . In such a limit the loop algebra acts like

$$L \cdot \Psi(\nabla) = \nabla(L) \cdot \Psi(\nabla) = \nabla_0(L) \cdot \Psi(\nabla) .$$

⁶such objects remain to be defined rigorously.

Since each loop L will generate an element $\nabla_0(L)$ in G the entire algebra of loops will reduce to a (sub) matrix algebra $M_N(\mathbb{C})$. In case ∇_0 equals the flat geometry, $\nabla_0(L) \equiv I$, the algebra appears to be abelian, simply \mathbb{C} . Further, it seems natural to 'average' the construction over the whole manifold since the choice of a basepoint x_0 breaks part of the diffeomorphism invariance. This amounts to multiplying the matrix algebra with the function algebra of the manifold. The result is an almost commutative algebra

$$C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C}) , \quad (20)$$

or a sub-algebra thereof. With the connection ∇_0 we are provided with a differential structure on the manifold and thus with a Dirac operator and a corresponding Hilbert space. Thus, these heuristic arguments show that an almost commutative algebra may emerge in the classical limit and that the matrix part of the algebra is related to the group algebra of the Lorentz group.

Recall that an algebra of the general form (20) provides, within the framework of noncommutative geometry, the basis for a Yang-Mills-Higgs model coupled to gravity. This is encouraging and provides us with the hope that low-energy physics may be recovered in the classical limit of pure quantum gravity.

The interesting question of what a semiclassical state might look like remains.

An interpretation of the Dirac operator

A connection is determined by holonomies along loops. In the projective system described here we consider first a finite number of loops and a connection is thus described 'coarse-grained' by assigning group elements to each of the finitely many elementary loops (or edges in the corresponding graph). The Dirac operator takes the derivative on each of these copies of the group G and throws it into the Clifford bundle. In this way the Dirac operator resembles a functional derivation operator.

We interpret this Dirac operator as intrinsically 'quantum' since it bears some resemblance to a canonical conjugate of the connection. Heuristically, we write

$$D \sim \frac{\delta}{\delta \nabla} \quad (21)$$

and

$$L \rightarrow \text{Hol}(L, \nabla) \sim 1 + \nabla \quad (22)$$

due to the loops L 's relation to the holonomy map. From (21) and (22) we obtain the non-vanishing commutator

$$[D, L] \sim [\frac{\delta}{\delta \nabla}, \nabla] \neq 0 ,$$

which shows a resemblance to a commutation relation of canonical conjugate variables. This means that the Dirac operator is intimately linked to the quantization of the functional space of connections.

5 Conclusion and outlook

Noncommutative geometry provides us with an exciting interpretation of the standard model as a gravitational theory. This formulation has many appealing features but fails to offer a quantization procedure compatible with the framework. In the introduction we argued that this problem is inevitable since a quantization procedure within this form would necessarily include quantum gravity. Loop Quantum Gravity provides us with ideas on background independent quantization of gravity but lacks, on the other hand, any notion of unification. Here we suggest that there might exist an intersection of the two: we study a noncommutative algebra of space-time holonomy loops which is interpreted as an algebra of functions over a space of connections. The space is described as a projective system of certain manifolds (Lie-groups) on which spectral triples are constructed. The whole construction is countable up to (extended) diffeomorphisms. The inner product in the emerging Hilbert space is interpreted as a functional integral. Also, the Dirac operator resembles a functional derivation on the space of connections. Finally, we provide heuristic arguments that a classical limit might contain an almost commutative geometry which forms the basis of Yang-Mills-Higgs models.

Many open issues remain. Let us here mention the most important points: First of all, it is still unclear whether a consistent embedding of Hilbert spaces exist. In [2] we presented a consistent construction for non-intersecting loops. For intersecting loops, however, the problem remains how to construct a metric which is compatible with all projections between graphs.

Next, it would be natural to consider a construction for non-compact groups, preferably $SO(3,1)$. This, however, presents difficulties involving both embedding problems and problems of constructing a spectral triple on non-compact spaces [18].

Finally, it remains to clarify whether the emergent construction satisfy basic requirements of a spectral triple. In particular, we find that the Dirac operator, in the case of non-intersecting loops, has infinite dimensional eigenspaces. This points in the direction of semifinite spectral triples [19].

Assuming these difficulties can be resolved, an interesting question to address is that of a semiclassical limit. One may speculate whether perturbations around a classical limit can generate some kind of 'quantization' of the fields which presumable emerge.

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